

On the integral of products of higher-order Bernoulli and Euler polynomials

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Abstract

In this paper, we derive a formula on the integral of products of the higher-order Euler polynomials. By the same way, similar relations are obtained for l higher-order Bernoulli polynomials and r higher-order Euler polynomials. Moreover, we establish the connection between the results and the generalized Dedekind sums and Hardy–Berndt sums. Finally, the Laplace transform of Euler polynomials is given.

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1 Introduction

The classical Bernoulli polynomials $B_m(x)$ and Euler polynomials $E_m(x)$ are usually defined by means of the following generating functions:

$$\frac{ue^{uz}}{e^u - 1} = \sum_{m=0}^{\infty} B_m(z) \frac{u^m}{m!} \quad (|u| < 2\pi) \quad \text{and} \quad \frac{2e^{uz}}{e^u + 1} = \sum_{m=0}^{\infty} E_m(z) \frac{u^m}{m!} \quad (|u| < 2\pi). \quad (1)$$

In particular, the rational numbers $B_m = B_m(0)$ and integers $E_m = 2^m E_m(1/2)$ are called classical Bernoulli numbers and Euler numbers, respectively.

As is well known, the classical Bernoulli and Euler polynomials play important roles in different areas of mathematics such as number theory, combinatorics, special functions and analysis.

This paper is primarily concerned with the higher-order Bernoulli and Euler polynomials. We derive a formula for the integral having r higher-order Euler polynomials and also for l higher-order Bernoulli and r higher-order Euler polynomials. The result is the corresponding generalization of some formulae discovered by Agoh and Dilcher [1], Hu et al [9] and of course [4, 7, 12, 14, 15, 16]. From our formula, we establish the connection between the sums of products of Euler (and Bernoulli and Euler) polynomials and the reciprocity formula for generalized Dedekind (and Hardy–Berndt) sums, motivated by Dağlı and Can [6].

We now turn to the higher-order Bernoulli and higher-order Euler polynomials. The higher-order Bernoulli polynomials $B_m^{(\alpha)}(x)$ and higher-order Euler polynomials $E_m^{(\alpha)}(x)$, each of degree m in x and in α , are defined by means of the generating functions [15]

$$\left(\frac{u}{e^u - 1}\right)^\alpha e^{uz} = \sum_{m=0}^{\infty} B_m^{(\alpha)}(z) \frac{u^m}{m!} \quad \text{and} \quad \left(\frac{2}{e^u + 1}\right)^\alpha e^{uz} = \sum_{m=0}^{\infty} E_m^{(\alpha)}(z) \frac{u^m}{m!},$$

respectively. For $\alpha = 1$, we have $B_m^{(1)}(z) = B_m(z)$ and $E_m^{(1)}(z) = E_m(z)$. They possess the differential property

$$\frac{d}{dz} B_m^{(\alpha)}(z) = m B_{m-1}^{(\alpha)}(z), \quad \frac{d}{dz} E_m^{(\alpha)}(z) = m E_{m-1}^{(\alpha)}(z) \quad (2)$$

and reciprocal relations

$$B_m^{(\alpha)}(\alpha - z) = (-1)^m B_m^{(\alpha)}(z), \quad E_m^{(\alpha)}(\alpha - z) = (-1)^m E_m^{(\alpha)}(z) \quad (3)$$

which imply $B_m^{(\alpha)}(\alpha/2) = 0$ and $E_m^{(\alpha)}(\alpha/2) = 0$ for odd m .

Also, we need the following expression of the Euler polynomials in terms of Bernoulli polynomials

$$E_n(x) = \frac{2}{n+1} \{B_{n+1}(x) - 2^{n+1}B_{n+1}(x/2)\} \quad (4)$$

for $n \geq 0$.

We summarize this study as follows: we firstly obtain several convolution formulas for higher-order Bernoulli and Euler polynomials applying the generating function methods, motivated by [5]. We also derive a formula for the integral having higher-order Euler polynomials. By this, we extend the result of Hu et al [9] and Liu et al [10]. By the same way, similar relations are obtained for l higher-order Bernoulli polynomials and r higher-order Euler polynomials, as well. Moreover, we establish the connection between the results and the reciprocity formulas for generalized Dedekind sums $T_r(c, d)$ and generalized Hardy-Berndt sums $s_{3,r}(c, d)$ and $s_{4,r}(c, d)$.

2 Convolutions of higher-order Bernoulli and Euler polynomials

In this section, we obtain some convolutions involving higher-order Bernoulli and Euler polynomials we will use in the next section.

Differentiating the generating function of higher-order Euler polynomials as follows

$$\begin{aligned} \frac{d}{du} \left(\left(\frac{2}{e^u + 1} \right)^n e^{uz} \right) &= \frac{2^n z e^{uz}}{(e^u + 1)^n} - \frac{n 2^n e^{u(z+1)}}{(e^u + 1)^{n+1}} \\ &= 2^n \frac{d}{du} \frac{e^{uz}}{(e^u + 1)^n}, \end{aligned}$$

we have

$$\frac{n 2^{n+1} e^{u(z+1)}}{(e^u + 1)^{n+1}} = \frac{2^{n+1} z e^{uz}}{(e^u + 1)^n} - 2^{n+1} \frac{d}{du} \frac{e^{uz}}{(e^u + 1)^n}.$$

Taking $z = x + y - 1$ and $n = \beta + \gamma - 1$ leads

$$\begin{aligned} \frac{n 2^{n+1} e^{u(z+1)}}{(e^u + 1)^{n+1}} &= \left(\sum_{m=0}^{\infty} E_m^{(\beta)}(x) \frac{u^m}{m!} \right) \left(\sum_{k=0}^{\infty} E_k^{(\gamma)}(y) \frac{u^k}{k!} \right) \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^m \binom{m}{k} E_k^{(\beta)}(x) E_{m-k}^{(\gamma)}(y) \frac{u^m}{m!}, \\ \frac{2^{n+1} z e^{uz}}{(e^u + 1)^n} &= 2(x + y - 1) \sum_{m=0}^{\infty} E_m^{(\beta+\gamma-1)}(x + y - 1) \frac{u^m}{m!} \end{aligned}$$

and

$$2^{n+1} \frac{d}{du} \frac{e^{uz}}{(e^u + 1)^n} = 2 \sum_{m=0}^{\infty} E_{m+1}^{(\beta+\gamma-1)}(x + y - 1) \frac{u^m}{m!}.$$

By equating the coefficients of $\frac{u^m}{m!}$, we get the convolution formula

$$\sum_{k=0}^m \binom{m}{k} E_k^{(\beta)}(x) E_{m-k}^{(\gamma)}(y) = 2(x + y - 1) E_m^{(\beta+\gamma-1)}(x + y - 1) - 2 E_{m+1}^{(\beta+\gamma-1)}(x + y - 1). \quad (5)$$

Similarly, for higher-order Bernoulli polynomials, we obtain

$$\begin{aligned} \sum_{k=0}^m \binom{m}{k} B_k^{(\beta)}(x) B_{m-k}^{(\gamma)}(y) \\ = (x + y - 1) m B_{m-1}^{(\beta+\gamma-1)}(x + y - 1) + (\gamma + \beta - 1 - m) B_m^{(\beta+\gamma-1)}(x + y - 1). \end{aligned}$$

From the generating functions of the higher-order Bernoulli and Euler polynomials, we can write

$$\left(\frac{u}{e^u - 1}\right)^n e^{xu} \left(\frac{2}{e^u + 1}\right)^n e^{yu} = \left(\frac{2u}{e^{2u} - 1}\right)^n e^{u(x+y)}.$$

Thus, similar arguments give the following convolution formula

$$\sum_{k=0}^m \binom{m}{k} B_{m-k}^{(n)}(x) E_k^{(n)}(y) = 2^m B_m^{(n)}\left(\frac{x+y}{2}\right). \quad (6)$$

3 Integral of products of higher-order Bernoulli and Euler polynomials

This section is devoted to obtain the integral of products of r higher-order Euler polynomials. Also, we derive a formula for the integral of products of l higher-order Bernoulli polynomials and r higher-order Euler polynomials. Furthermore, we relate these results to the reciprocity formulas for generalized Dedekind sums $T_r(c, d)$ and Hardy-Berndt sums $s_{3,r}(c, d)$ and $s_{4,r}(c, d)$.

3.1 Euler polynomials

Theorem 3.1 *Let $b_1, \dots, b_r, y_1, \dots, y_r$ be arbitrary real numbers with $b_s \neq 0, 1 \leq s \leq r$, and*

$$\begin{aligned} \hat{I}_{n_1, \dots, n_r}(x; b; y) &= \hat{I}_{n_1, \dots, n_r}(x; b_1, \dots, b_r; y_1, \dots, y_r) \\ &= \frac{1}{n_1! \cdots n_r!} \int_0^x \prod_{s=1}^r E_{n_s}^{(\alpha_s)}(b_s z + y_s) dz, \\ \hat{C}_{n_1, \dots, n_r}(x; b; y) &= \hat{C}_{n_1, \dots, n_r}(x; b_1, \dots, b_r; y_1, \dots, y_r) \\ &= \frac{1}{n_1! \cdots n_r!} \left(\prod_{s=1}^r E_{n_s}^{(\alpha_s)}(b_s x + y_s) - \prod_{s=1}^r E_{n_s}^{(\alpha_s)}(y_s) \right) \end{aligned}$$

Then

$$\begin{aligned} \hat{I}_{n_1, \dots, n_r}(x; b; y) &= \sum_{a=0}^{\mu} (-1)^a \sum_{j_1 + \dots + j_{r-1} = a} \binom{a}{j_1, \dots, j_{r-1}} b_1^{j_1} \cdots b_{r-1}^{j_{r-1}} b_r^{-a-1} \hat{C}_{n_1-j_1, \dots, n_{r-1}-j_{r-1}, n_r+a+1}(x; b; y) \\ &\quad + \frac{(-1)^{\mu+1}}{(n+\mu+1)!} \int_0^x \left(\prod_{s=1}^{r-1} E_{n_s}^{(\alpha_s)}(b_s z + y_s) \right)^{(\mu+1)} E_{n_r+\mu+1}^{(\alpha_r)}(b_r z + y_r) dz, \end{aligned}$$

where $\binom{\mu}{n_1, \dots, n_r}$ are the multinomial coefficients defined by

$$\binom{\mu}{n_1, \dots, n_r} = \frac{\mu!}{n_1! \cdots n_r!}, \quad n_1 + \cdots + n_r = \mu \text{ and } n_1, \dots, n_r \geq 0.$$

In particular if $\mu = n_1 + \cdots + n_{r-1}$, we have

$$\begin{aligned} \hat{I}_{n_1, \dots, n_r}(x; b; y) &= \sum_{a=0}^{\mu} (-1)^a \sum_{j_1 + \dots + j_{r-1} = a} \binom{a}{j_1, \dots, j_{r-1}} b_1^{j_1} \cdots b_{r-1}^{j_{r-1}} \\ &\quad \times b_r^{-a-1} \hat{C}_{n_1-j_1, \dots, n_{r-1}-j_{r-1}, n_r+a+1}(x; b; y). \end{aligned} \quad (7)$$

Proof. Let

$$f(z) = E_{n_1}^{(\alpha_1)}(b_1 z + y_1) \cdots E_{n_{r-1}}^{(\alpha_{r-1})}(b_{r-1} z + y_{r-1}).$$

Then

$$\begin{aligned} & \frac{1}{n_r!} \int_0^x f(z) E_{n_r}^{(\alpha_r)}(b_r z + y_r) dz \\ &= \left[\frac{1}{b_r (n_r + 1)!} f(z) E_{n_r+1}^{(\alpha_r)}(b_r z + y_r) \right]_0^x - \frac{1}{(n_r + 1)!} \int_0^x f'(z) E_{n_r+1}^{(\alpha_r)}(b_r z + y_r) dz. \end{aligned}$$

Using μ additional integrations by parts, we find that

$$\begin{aligned} \frac{1}{n_r!} \int_0^x f(z) E_{n_r}^{(\alpha_r)}(b_r z + y_r) dz &= \sum_{a=0}^{\mu} \frac{(-1)^a}{(n_r + a + 1)!} \left[f^{(a)}(z) E_{n_r+a+1}^{(\alpha_r)}(b_r z + y_r) \right]_0^x \\ &\quad + \frac{(-1)^{\mu+1}}{(n_r + \mu + 1)!} \int_0^x f^{(\mu+1)}(z) E_{n_r+\mu+1}^{(\alpha_r)}(b_r z + y_r) dz. \end{aligned} \quad (8)$$

Using the property of derivative

$$(f_1(z) \cdots f_m(z))^{(a)} = \sum_{j_1 + \cdots + j_m = a} \binom{a}{j_1, \dots, j_m} f_1^{(j_1)}(z) \cdots f_m^{(j_m)}(z),$$

and (2), we get the desired result. ■

Setting $x = 1$ and $b_s = \alpha_s - 2y_s$ with $y_s \neq \alpha_s/2$, $1 \leq s \leq r$, in (7) we have

$$\begin{aligned} & \widehat{I}_{n_1, \dots, n_r}(1; \alpha_1 - 2y_1, \dots, \alpha_r - 2y_r; y_1, \dots, y_r) \\ &= \sum_{a=0}^{n_1 + \cdots + n_{r-1}} (-1)^a \sum_{j_1 + \cdots + j_{r-1} = a} \binom{a}{j_1, \dots, j_{r-1}} \frac{((-1)^{n_1 + \cdots + n_{r-1} + 1} - 1)}{(n_1 - j_1)! \cdots (n_r + a + 1)!} \\ &\quad \times b_1^{j_1} \cdots b_{r-1}^{j_{r-1}} b_r^{-a-1} E_{n_1-j_1}^{(\alpha_1)}(y_1) \cdots E_{n_{r-1}-j_{r-1}}^{(\alpha_{r-1})}(y_{r-1}) E_{n_r+a+1}^{(\alpha_r)}(y_r) \end{aligned}$$

since $E_{n_s-j_s}^{(\alpha_s)}(b_s - y_s) = E_{n_s-j_s}^{(\alpha_s)}(\alpha_s - y_s) = (-1)^{n_s-j_s} E_{n_s-j_s}^{(\alpha_s)}(y_s)$ and $j_1 + \cdots + j_{r-1} = a$. Therefore, if $n_1 + \cdots + n_r + 1$ is even, then

$$\widehat{I}_{n_1, \dots, n_r}(1; \alpha_1 - 2y_1, \dots, \alpha_r - 2y_r; y_1, \dots, y_r) = 0,$$

and if $n_1 + \cdots + n_r + 1$ is odd, then

$$\begin{aligned} & \widehat{I}_{n_1, \dots, n_r}(1; \alpha_1 - 2y_1, \dots, \alpha_r - 2y_r; y_1, \dots, y_r) \\ &= -2 \sum_{a=0}^{n_1 + \cdots + n_{r-1}} (-1)^a \frac{(\alpha_r - 2y_r)^{-a-1}}{(n_r + a + 1)!} E_{n_r+a+1}^{(\alpha_r)}(y_r) \\ &\quad \times \sum_{j_1 + \cdots + j_{r-1} = a} \binom{a}{j_1, \dots, j_{r-1}} \prod_{s=1}^{r-1} \frac{(\alpha_s - 2y_s)^{j_s}}{(n_s - j_s)!} E_{n_s-j_s}^{(\alpha_s)}(y_s). \end{aligned}$$

For example, we have

$$\int_0^1 E_2^{(3)}(7z - 2) E_3^{(1/2)}\left(-\frac{3}{2}z + 1\right) E_{10}^{(5)}(4z + 1/2) dz = 0$$

and

$$\frac{1}{2!10!} \int_0^1 E_2^{(3)}(3z) E_{10}^{(5)}(-3z+4) dz = \frac{2}{3} \sum_{a=0}^2 \frac{E_{2-a}^{(3)}(0)}{(2-a)!} \frac{E_{11+a}^{(5)}(4)}{(11+a)!}.$$

It is seen from the definition of the integral $\widehat{I}_{n_1, \dots, n_r}(x; b; y)$ that the left-hand side of (7) is invariant under interchanging the order of the integrands. That is, for $r = 2$,

$$\begin{aligned} & \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} \\ & \times \left(E_{n-a}^{(\gamma)}(b_1 x + y_1) E_{m+a+1}^{(\beta)}(b_2 x + y_2) - E_{n-a}^{(\gamma)}(y_1) E_{m+a+1}^{(\beta)}(y_2) \right) \\ & = \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} \\ & \times \left(E_{m-a}^{(\gamma)}(b_2 x + y_2) E_{n+a+1}^{(\beta)}(b_1 x + y_1) - E_{m-a}^{(\gamma)}(y_2) E_{n+a+1}^{(\beta)}(y_1) \right). \end{aligned} \quad (9)$$

So, we may investigate the reciprocity relation for sums of products of higher-order Euler polynomials as follows: Let

$$\begin{aligned} T &:= \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} E_{n-a}^{(\gamma)}(y_1) E_{m+a+1}^{(\beta)}(y_2) \\ & - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} E_{m-a}^{(\gamma)}(y_2) E_{n+a+1}^{(\beta)}(y_1). \end{aligned}$$

We first rewrite this as

$$\begin{aligned} T &= \sum_{a=0}^n (-1)^{n-a} \binom{m+n+1}{a} b_1^{n-a} b_2^{a-n-1} E_a^{(\gamma)}(y_1) E_{m+n+1-a}^{(\beta)}(y_2) \\ & - \sum_{a=0}^m (-1)^{m-a} \binom{m+n+1}{a} b_2^{m-a} b_1^{a-m-1} E_a^{(\gamma)}(y_2) E_{m+n+1-a}^{(\beta)}(y_1). \end{aligned} \quad (10)$$

Without loss of generality we may assume that $n \geq m$; in this case we separate the sum from 0 to m and $m+1$ to n on the first summation in (10), and rewrite these as

$$\begin{aligned} & \sum_{a=0}^m (-1)^{n-a} \binom{m+n+1}{a} b_1^{n-a} b_2^{a-n-1} E_a^{(\gamma)}(y_1) E_{m+n+1-a}^{(\beta)}(y_2) \\ & = \sum_{a=n+1}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} b_1^{a-m-1} b_2^{m-a} E_{m+n+1-a}^{(\gamma)}(y_1) E_a^{(\beta)}(y_2) \end{aligned}$$

and

$$\begin{aligned} & \sum_{a=m+1}^n (-1)^{n-a} \binom{m+n+1}{a} b_1^{n-a} b_2^{a-n-1} E_a^{(\gamma)}(y_1) E_{m+n+1-a}^{(\beta)}(y_2) \\ & = \sum_{a=m+1}^n (-1)^{m+1-a} \binom{m+n+1}{a} b_1^{a-m-1} b_2^{m-a} E_{m+n+1-a}^{(\gamma)}(y_1) E_a^{(\beta)}(y_2). \end{aligned}$$

Thus, we have

$$\begin{aligned} T &= \frac{1}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} \\ & \times b_1^a b_2^{m+n+1-a} E_{m+n+1-a}^{(\gamma)}(y_1) E_a^{(\beta)}(y_2). \end{aligned} \quad (11)$$

Combining (9) and (11) gives the reciprocity relation for sums of products of higher-order Euler polynomials.

Corollary 3.2

$$\begin{aligned}
& \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} E_{n-a}^{(\gamma)}(b_1 x + y_1) E_{m+a+1}^{(\beta)}(b_2 x + y_2) \\
& - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} E_{m-a}^{(\gamma)}(b_2 x + y_2) E_{n+a+1}^{(\beta)}(b_1 x + y_1) \\
& = \frac{1}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} E_{m+n+1-a}^{(\gamma)}(y_1) E_a^{(\beta)}(y_2). \quad (12)
\end{aligned}$$

In particular for $y_1 = \gamma/2$, $y_2 = \beta/2$ and even $(m+n)$, the right-hand side of (12) vanishes.

Remark 3.3 Beginning from the left-hand side of (12) and using the arguments in the proof of (11), the right-hand side of (12) turns into

$$\begin{aligned}
& \frac{1}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} \\
& \times b_1^a b_2^{m+n+1-a} E_{m+n+1-a}^{(\gamma)}(b_1 x + y_1) E_a^{(\beta)}(b_2 x + y_2).
\end{aligned}$$

So it follows that for all x ,

$$\begin{aligned}
& \sum_{a=0}^{m+n+1} (-1)^a \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} E_{m+n+1-a}^{(\gamma)}(b_1 x + y_1) E_a^{(\beta)}(b_2 x + y_2) \\
& = \sum_{a=0}^{m+n+1} (-1)^a \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} E_{m+n+1-a}^{(\gamma)}(y_1) E_a^{(\beta)}(y_2).
\end{aligned}$$

- Let $b_1 = b_2 = 1$ in (12). Then the right-hand side becomes, with the use of (3),

$$(-1)^n T = \sum_{a=0}^{m+n+1} \binom{m+n+1}{a} E_{m+n+1-a}^{(\gamma)}(\gamma - y_1) E_a^{(\beta)}(y_2).$$

Now using (5) by taking $x = y_2$ and $y = \gamma - y_1$, (12) reduces to

$$\begin{aligned}
& \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} E_{n-a}^{(\gamma)}(x + y_1) E_{m+a+1}^{(\beta)}(x + y_2) \\
& - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} E_{m-a}^{(\gamma)}(x + y_2) E_{n+a+1}^{(\beta)}(x + y_1) \\
& = (-1)^n 2(y_2 - y_1 + \gamma - 1) E_{m+n+1}^{(\gamma+\beta-1)}(y_2 - y_1 + \gamma - 1) \\
& - (-1)^n 2E_{m+n+2}^{(\gamma+\beta-1)}(y_2 - y_1 + \gamma - 1). \quad (13)
\end{aligned}$$

- Setting $b_1 = 1$, $b_2 = -1$ and using (5), (12) becomes

$$\begin{aligned}
& \sum_{a=0}^n \binom{m+n+1}{n-a} E_{n-a}^{(\gamma)}(x + y_1) E_{m+a+1}^{(\beta)}(y_2 - x) \\
& + \sum_{a=0}^m \binom{m+n+1}{m-a} E_{m-a}^{(\gamma)}(y_2 - x) E_{n+a+1}^{(\beta)}(x + y_1) \\
& = 2(y_2 + y_1 - 1) E_{m+n+1}^{(\gamma+\beta-1)}(y_2 + y_1 - 1) - 2E_{m+n+2}^{(\gamma+\beta-1)}(y_2 + y_1 - 1).
\end{aligned}$$

- Set $\beta = \gamma = 1$, $b_1 = 2$ and $b_2 = -1$ in (12). In view of [5, Theorem 6], (12) becomes

$$\begin{aligned}
& \sum_{a=0}^n \binom{m+n+1}{n-a} 2^{m+1+a} E_{n-a}(2x+y_1) E_{m+a+1}(-x+y_2) \\
& + \sum_{a=0}^m \binom{m+n+1}{m-a} 2^{m-a} E_{m-a}(-x+y_2) E_{n+a+1}(2x+y_1) \\
& = \sum_{a=0}^{m+n+1} \binom{m+n+1}{a} 2^a E_a(y_2) E_{m+n+1-a}(y_1) \\
& = E_{m+n+1}(2y_2+y_1) + 2^{m+n+1} E_{m+n+1} \left(\frac{2y_2+y_1}{2} \right) \\
& \quad - 2^{m+n+1} E_{m+n+1} \left(\frac{2y_2+y_1+1}{2} \right).
\end{aligned}$$

- Let $\gamma = \beta = 1$ and $y_1 = y_2 = 0$ in (12). Then,

$$\begin{aligned}
& \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} E_{n-a}(b_1 x) E_{m+a+1}(b_2 x) \\
& - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} E_{m-a}(b_2 x) E_{n+a+1}(b_1 x) \\
& = \frac{1}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} E_{m+n+1-a}(0) E_a(0). \tag{14}
\end{aligned}$$

From the property $B_{2n+1}(0) = 0$, $n \geq 1$ and (4) for $x = 0$, we have $(-1)^a E_a(0) = -E_a(0)$ for $a > 0$. Then, the right-hand side of (14) can be written

$$\begin{aligned}
T &= \frac{(-1)^m}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} \\
&\quad \times E_{m+n+1-a}(0) E_a(0) - 2 \frac{(-b_2)^m}{b_1^{m+1}} E_{m+n+1}(0). \tag{15}
\end{aligned}$$

Remark 3.4 Kim and Son [11] proved the reciprocity formula for generalized Dedekind sums $T_r(c, d)$ as

$$cd^r T_r(c, d) + dc^r T_r(d, c) = -\frac{1}{2} \sum_{a=0}^r \binom{r}{a} d^{a-1} c^{r-1-a} \overline{E}_a(0) \overline{E}_{r-a}(0) + \overline{E}_{r+1}(0), \tag{16}$$

where $T_r(d, c)$ is defined by

$$T_r(c, d) = \sum_{j=0}^{|d|-1} (-1)^j \overline{E}_1 \left(\frac{j}{d} \right) \overline{E}_r \left(\frac{cj}{d} \right)$$

in which

$$\begin{aligned}
\overline{E}_r(x) &= E_r(x), \quad 0 \leq x < 1, \\
\overline{E}_r(x+p) &= (-1)^p \overline{E}_r(x), \quad p \in \mathbb{Z}.
\end{aligned}$$

It is seen from (15) and (16) that the reciprocity formula of the generalized Dedekind sum $T_r(c, d)$ can be written in terms of the reciprocity relation of Euler polynomials.

Now, let us give the Laplace transform of $\overline{E}_n(tu)$ by applying (8).

Example 3.5 Let $\operatorname{Re}(s) > 0$ and $|s/t| < \pi$. Setting $f(u) = e^{-su}$ and $\overline{E}_n(tu)$ instead of $E_{n_r}^{(\alpha_r)}(u)$ in (8) gives

$$\begin{aligned} \frac{1}{n!} \int_0^x e^{-su} \overline{E}_n(tu) du &= \sum_{a=0}^{\mu} \frac{s^a t^{-a-1}}{(n+a+1)!} \{e^{-sx} \overline{E}_{n+a+1}(tx) - \overline{E}_{n+a+1}(0)\} \\ &\quad + \left(\frac{s}{t}\right)^{\mu+1} \frac{1}{(n+\mu+1)!} \int_0^x e^{-su} \overline{E}_{n+\mu+1}(tu) du. \end{aligned} \quad (17)$$

Since the function $\overline{E}_m(u) = (-1)^{[u]} E(u - [u])$ is bounded, the integrals in (17) converge absolutely and $e^{-sx} \overline{E}_{n+a+1}(tx)$ tends to 0 as $x \rightarrow \infty$. Then, letting $x \rightarrow \infty$, we have

$$\begin{aligned} \frac{1}{n!} \int_0^{\infty} e^{-su} \overline{E}_n(tu) du &= -\frac{t^n}{s^{n+1}} \sum_{a=0}^{\mu} \frac{E_{n+a+1}(0)}{(n+a+1)!} \frac{s^{n+a+1}}{t^{n+a+1}} \\ &\quad + \left(\frac{s}{t}\right)^{\mu+1} \frac{1}{(n+\mu+1)!} \int_0^{\infty} e^{-su} \overline{E}_{n+\mu+1}(tu) du. \end{aligned} \quad (18)$$

From (1) the sum in (18) converges absolutely for $|s/t| < \pi$ as $\mu \rightarrow \infty$. Also the sequence of the functions (in u) $s^\mu \overline{E}_\mu(tu) / \mu! t^\mu$ converges uniformly to 0 for $|s/t| < \pi$. Thus, letting $\mu \rightarrow \infty$ and using (1), we obtain the Laplace transform of $\overline{E}_n(tu)$

$$\begin{aligned} \frac{1}{n!} \int_0^{\infty} e^{-su} \overline{E}_n(tu) du &= -\frac{t^n}{s^{n+1}} \sum_{a=0}^{\infty} \frac{\overline{E}_{n+a+1}(0)}{(n+a+1)!} \frac{s^{n+a+1}}{t^{n+a+1}} \\ &= \frac{t^n}{s^{n+1}} \left(\sum_{a=0}^n \frac{E_a(0)}{a!} \frac{s^a}{t^a} - \sum_{a=0}^{\infty} \frac{E_a(0)}{a!} \frac{s^a}{t^a} \right) \\ &= \frac{1}{s} \sum_{a=0}^n \frac{E_a(0)}{a!} \left(\frac{t}{s} \right)^{n-a} - \frac{t^n}{s^{n+1}} \frac{2}{e^{s/t} + 1}. \end{aligned} \quad (19)$$

Note that for $t = 1$, (19) coincides with [13, eq. (64)]. Differentiating m times both sides of (19) with respect to s , we have

$$\frac{(-1)^m}{n!} \int_0^{\infty} u^m e^{-su} \overline{E}_n(tu) du = \frac{d^m}{ds^m} \left(\sum_{a=0}^n \frac{E_a(0)}{a!} t^{n-a} s^{a-n-1} - \frac{t^n}{s^{n+1}} \frac{2}{e^{s/t} + 1} \right).$$

3.2 Bernoulli and Euler polynomials

Theorem 3.6 Let b_s and y_s , $1 \leq s \leq l+r$ be arbitrary real numbers with $b_s \neq 0$. Let $N = n_1! \cdots n_l! m_1! \cdots m_r!$ and

$$\begin{aligned} J_{n_1, \dots, m_r}(x; b; y) &= J_{n_1, \dots, m_r}(x; b_1, \dots, b_{l+r}; y_1, \dots, y_{l+r}) \\ &= \frac{1}{N} \int_0^x \prod_{s=1}^l B_{n_s}^{(\gamma_s)}(b_s z + y_s) \prod_{i=1}^r E_{m_i}^{(\beta_i)}(b_{l+i} z + y_{l+i}) dz, \\ D_{n_1, \dots, m_r}(x; b; y) &= D_{n_1, \dots, m_r}(x; b_1, \dots, b_{l+r}; y_1, \dots, y_{l+r}) \\ &= \frac{1}{N} \prod_{s=1}^l B_{n_s}^{(\gamma_s)}(b_s x + y_s) \prod_{i=1}^r E_{m_i}^{(\beta_i)}(b_{l+i} x + y_{l+i}) \\ &\quad - \frac{1}{N} \prod_{s=1}^l B_{n_s}^{(\gamma_s)}(y_s) \prod_{i=1}^r E_{m_i}^{(\beta_i)}(y_{l+i}). \end{aligned}$$

Then, for $\mu = n_1 + \dots + n_l + m_1 + \dots + m_{r-1}$,

$$J_{n_1, \dots, m_r}(x; b; y) = \sum_{a=0}^{\mu} (-1)^a \sum_{j_1 + \dots + j_{l+r-1} = a} \binom{a}{j_1, \dots, j_{l+r-1}} \\ \times b_1^{j_1} \dots b_{l+r-1}^{j_{l+r-1}} b_{l+r}^{-a-1} D_{n_1-j_1, \dots, m_{r-1}-j_{l+r-1}, m_r+a+1}(x; b; y). \quad (20)$$

Proof. The proof can be obtained by using the arguments in the proof of Theorem (3.1). ■

In order to obtain the reciprocity relation for sums of products of higher-order Bernoulli and Euler polynomials, similar to T , we define

$$T_1 := \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} B_{n-a}^{(\gamma)}(y_1) E_{m+a+1}^{(\beta)}(y_2) \\ - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} E_{m-a}^{(\gamma)}(y_2) B_{n+a+1}^{(\beta)}(y_1).$$

Similarly, we have

$$T_1 = \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} B_{n-a}^{(\gamma)}(b_1 x + y_1) E_{m+a+1}^{(\beta)}(b_2 x + y_2) \\ - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} E_{m-a}^{(\gamma)}(b_2 x + y_2) B_{n+a+1}^{(\beta)}(b_1 x + y_1) \\ = \frac{1}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} \\ \times b_1^a b_2^{m+n+1-a} E_a^{(\beta)}(y_2) B_{m+n+1-a}^{(\gamma)}(y_1). \quad (21)$$

Notice that the right-hand side of (21) vanishes for $y_1 = \gamma/2$, $y_2 = \beta/2$ and even $m+n$.

- Setting $\beta = \gamma = 1$ and $b_1 = 2$, $b_2 = -1$ in (21), we get

$$2^{m+1} T_1 = - \sum_{a=0}^{m+n+1} \binom{m+n+1}{a} 2^a E_a(y_2) B_{m+n+1-a}(y_1) \\ = -B_{m+n+1}(2y_2 + y_1) + 2^{m+n-1} (m+n+1) E_{m+n} \left(\frac{2y_2 + y_1 + 1}{2} \right) \\ - 2^{m+n-1} (m+n+1) E_{m+n} \left(\frac{2y_2 + y_1}{2} \right)$$

by [5, Theorem 10]. After similar manipulations to T , we have for $\gamma = \beta$ and $b_2 = b_1 = 1$

$$(-1)^n T_1 = \sum_{a=0}^{m+n+1} \binom{m+n+1}{a} B_{m+n+1-a}^{(\gamma)}(\gamma - y_1) E_a^{(\gamma)}(y_2).$$

In view of (6) for $x = \gamma - y_1$, $y = y_2$, we get

$$T_1 = (-1)^n 2^{m+n+1} B_{m+n+1}^{(\gamma)} \left(\frac{\gamma - y_1 + y_2}{2} \right).$$

- On the other hand, for $y_1 = y_2 = 0$ and $\gamma = \beta = 1$, T_1 can be written as

$$T_1 = \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} B_{n-a}(0) E_{m+a+1}(0) \\ - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} E_{m-a}(0) B_{n+a+1}(0) \\ = \frac{(-1)^{m+1}}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^a \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} B_{m+n+1-a}(0) E_a(0).$$

Using (4) for $x = 0$, we get

$$T_1 = \frac{(-1)^{m+1}}{b_1^{m+1}b_2^{n+1}} \sum_{a=1}^{m+n+2} (-1)^{a-1} \binom{m+n+1}{a-1} \\ \times b_1^{a-1}b_2^{m+n+2-a} B_{m+n+2-a} \frac{2}{a} (1-2^a) B_a.$$

Therefore, we have

$$\begin{aligned} & \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} B_{n-a} E_{m+a+1}(0) \\ & - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} E_{m-a}(0) B_{n+a+1} \\ & = \frac{(-1)^m}{b_1^{m+2}b_2^{n+1}} \frac{2}{m+n+2} \sum_{a=1}^{m+n+2} (-1)^a \binom{m+n+2}{a} \\ & \times b_1^a b_2^{m+n+2-a} (1-2^a) B_{m+n+2-a} B_a. \end{aligned} \quad (22)$$

Remark 3.7 Observe that the sum on the right-hand side of (22) is the reciprocity formula for the Hardy–Berndt sums $s_{3,r}(c, d)$ and $s_{4,r}(c, d)$ given by [3]

$$\begin{aligned} & (r+1) (cd^r s_{3,r}(c, d) - 2^{-2} d (2c)^r s_{4,r}(d, c)) \\ & = 2 \sum_{a=1}^{r+1} \binom{r+1}{a} (-1)^a c^a d^{r+1-a} (1-2^a) B_a B_{r+1-a}, \end{aligned} \quad (23)$$

where d and r are odd and

$$s_{3,r}(c, d) = \sum_{j=1}^{d-1} (-1)^j \overline{B}_r \left(\frac{cj}{d} \right), \quad s_{4,r}(c, d) = -4 \sum_{j=1}^{d-1} \overline{B}_r \left(\frac{cj}{2d} \right).$$

Thus, the reciprocity formulas given by (22) and (23) can be associated as

$$\begin{aligned} & \sum_{a=0}^n (-1)^{m-a} \binom{m+n+1}{n-a} b_1^{m+2+a} b_2^{n-a} B_{n-a} E_{m+a+1}(0) \\ & - \sum_{a=0}^m (-1)^{m-a} \binom{m+n+1}{m-a} b_2^{n+1+a} b_1^{m+1-a} E_{m-a}(0) B_{n+a+1} \\ & = b_1 b_2^r s_{3,r}(b_1, b_2) - 2^{-2} b_2 (2b_1)^r s_{4,r}(b_2, b_1) \\ & = \frac{2}{r+1} \sum_{a=1}^{r+1} (-1)^a \binom{r+1}{a} b_1^a b_2^{r+1-a} (1-2^a) B_{r+1-a} B_a \end{aligned}$$

for odd integers $r = (m+n+1)$ and b_2 .

From this relationship, (22) can be evaluated for some special cases. Since $s_{3,r}(d, 1) = 0$ and $s_{4,r}(d, 1) = 0$, we have

$$\begin{aligned} & \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b^{-a-1} B_{n-a} E_{m+a+1}(0) \\ & - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b^a E_{m-a}(0) B_{n+a+1} \\ & = (-1)^m b^m s_{3,m+n+1}(1, b) \end{aligned}$$

for odd integers $(m + n + 1)$ and b , and

$$\begin{aligned} & \sum_{a=0}^n (-1)^a \binom{m+n+1}{n-a} b^a B_{n-a} E_{m+a+1}(0) \\ & - \sum_{a=0}^m (-1)^a \binom{m+n+1}{m-a} b^{-a-1} E_{m-a}(0) B_{n+a+1} \\ & = (-1)^{m+1} 2^{m+n-1} b^{n-1} s_{4,m+n+1}(1, b) \end{aligned}$$

for odd integer $(m + n + 1)$.

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